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# A STUDY OF A VAN DER POL EQUATION

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### Abstract

In this paper, we make a study of a dynamical system on the plane without periodic orbits in a domain on the plane. We use a Gasull's result and Dulac's criterion that give sufficient conditions for the non-existence of periodic orbits of dynamical systems in simply connected regions of the plane.

## 1. Introduction

It is important to make in differential equations the analysis of the periodic orbits that there are in a system in the plane. Certain systems have no limit cycles. For this should be considered: Bendixson's criteria, indices, invariant lines, and critical points (see [1, 5, 8, 9]). In this paper,

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we are interested in studying a system that have a periodic orbit but in a circular domain of radius one and center in the origin, there is not this limit cycle. We use the criterion of Bendixson-Dulac (see [4]) and paper of Gasull (see [2]).

**Theorem 1.1** (Bendixson-Dulac criterion)(see [6]). *Let* $f_1(x_1, x_2)$, $f_2(x_1, x_2)$, *and* $h(x_1, x_2)$ *be functions* $C^1$ *in a simply connected domain* $D \subset \mathbb{R}^2$ *such that* $\frac{\partial(f_1 h)}{\partial x_1} + \frac{\partial(f_2 h)}{\partial x_2}$ *does not change sign in D and vanishes at most on a set of measure zero. Then the system*

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2), \\ \dot{x}_2 = f_2(x_1, x_2), \quad (x_1, x_2) \in D, \end{cases} \tag{1.1}$$

*does not have periodic orbits in D.*

According to this criterion, to rule out the existence of periodic orbits of the system (1.1) in a simply connected region $D$, we need to find a function $h(x_1, x_2)$ that satisfies the conditions of the theorem of Bendixson-Dulac, such function $h$ is called a Dulac function. In Saez and Szanto [7] was constructed Lyapunov functions by using Dulac functions to assure the non-existence of periodic orbits. Our goal is to study a dynamical system on the plane that not have periodic orbits in a circular domain of radius one and center in the origin.

## 2. Method to Obtain Dulac Functions

A Dulac function for the system (1.1) satisfies the equation:

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\left(c(x_1, x_2) - \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right) \tag{2.1}$$

(see [6]).

**Theorem 2.1** (See [6]). *For the system of differential equations* (1.1), *a solution h of the associated system* (2.1) (*for some function c which does not change sign and vanishes only on a subset of measure zero*) *is a Dulac function for* (1.1) *in any simply connected region A contained in* $D \backslash \{h^{-1}(0)\}$.

**Theorem 2.2** (See [6]). *For the system of differential equations* (1.1), *if* (2.1) (*for some function c which does not change of sign and subset of measure zero*) *has a solution h on D such that h does not change sign and vanishes only on a subset of measure zero*, *then h is a Dulac function for* (1.1) *on D*.

**Theorem 2.3** (See [2]). *Assume that there exist a real number s and an analytic function h in* $R^2$ *such that*

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\left(c(x_1, x_2) - s\left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right)\right), \tag{2.2}$$

*does not change sign in an open region* $W \subset R^2$ *with regular boundary and vanishes only in a null measure Lebesgue set*. *Then the limit cycles of system* (1.1) *are either totally contained in* $\mathfrak{h}_0 := \{h = 0\}$, *or do not intersect* $\mathfrak{h}_0$. *Moreover*, *the number N of limit cycles that do not intersect* $\mathfrak{h}_0$ *satisfies* $N = 0$ *if* $s = 0$.

## 3. Main Result

**Theorem 3.1.** *Let* $f(x_1)$, $g(x_1)$ *be functions* $C^1$ *in a simply connected domain* $D = \{h \leq 0\} \subset \mathbb{R}^2$, *where* $h(x_1, x_2) = \psi(x_1) + ax_2^2 + bx_2 + c$, *where* $\psi(x_1)$ *is a function* $C^1$ *in* $\mathbb{R}$, $a, b, c \in \mathbb{R}$ *with the following conditions* $(2ag(x_1) + bf(x_1) - \psi'(x_1))^2 - 8abf(x_1)g(x_1) \leq 0$ *and* $b^2 - 4a(\psi(x_1) + c) \geq 0$. *Then the system*

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -f(x_1)x_2 - g(x_1), \end{cases} \tag{3.1}$$

*does not have periodic orbits in D*.

**Proof.** Applying Theorem 2.3 to the Liénard system (3.1)(this system has critical points on the axis $x_1$ and on the zeros of $g(x_1)$). From (2.2), we see functions $f(x_1)$ and $g(x_1)$ and values of $s$ satisfy the equation

$$x_2(h_{x_1}) - (g(x_1) + f(x_1)x_2)(h_{x_2}) = h(c(x_1, x_2) + sf(x_1)), \tag{3.2}$$

for some $h$, $c(x_1, x_2)$ with $hc$ does not change of sign (except in a set of measure 0). Obviously, $h$ is a Dulac function in certain cases. We propose (instead of try to solve Equation (3.2)) the function $h = \psi(x_1) + ax_2^2 + bx_2 + c$ for adequate $\psi$ such that $h$ has a closed curve of level 0. When $h = 0$, we have $x_2 = \dfrac{-b \pm \sqrt{b^2 - 4a(\psi(x_1) + c)}}{2a}$. Then $b^2 - 4a(\psi(x_1) + c) \geq 0$. We try to find the domain for which the system does not have periodic orbits. We have $h_{x_1} = \psi'(x_1)$, $h_{x_2} = 2ax_2 + b$. So, we have

$$af(x_1)(s + 2)x_2^2 + (2ag(x_1) + bf(x_1) + bsf(x_1) - \psi'(x_1))x_2$$
$$+ bg(x_1) + sf(x_1)c + sf(x_1)\psi, \tag{3.3}$$

which does not change sign and it vanishes only in a null measure Lebesgue subset. Making $s = 0$ (this system would not have periodic orbits inside the domain with boundary $h = 0$), we get that $2af(x_1)x_2^2 + (2ag(x_1) + bf(x_1) - \psi'(x_1))x_2 + bg(x_1)$, which does not change sign and it vanishes only in a null measure Lebesgue subset, if the discriminant is $(2ag(x_1) + bf(x_1) - \psi'(x_1))^2 - 8abf(x_1)g(x_1) \leq 0$.

□

**Example 3.2.** Consider the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\varepsilon(x_1^2 - 1)x_2 - x_1. \end{cases} \tag{3.4}$$

Taking $h(x_1, x_2) = x_1^2 + x_2^2 - 1$, we obtain that the associated equation given in (2.2) is $hc(x_1, x_2) = -\varepsilon(x_1^2 - 1)(sx_1^2 + (2 + s)x_2^2 - s)$. Choosing $s = 0$, we get that $hc(x_1, x_2) = -2\varepsilon(x_1^2 - 1)x_2^2$. So, for $\varepsilon \neq 0$, this function does not change sign and it is zero only at $x_1 = \pm 1$, $x_2 = 0$ in $D = \{x_1^2 + x_2^2 \leq 1\}$. The system does not contain periodic orbits in $D$. By (3.4), we have $\ddot{x}_1 + \varepsilon(x_1^2 - 1)\dot{x}_1 + x_1 = 0$. This equation is generalized by $\ddot{x}_1 + f(x_1)\dot{x}_1 + g(x_1) = 0$. The last equation was studied by Liénard in [3]. This equation models mechanical systems and also it models resistor-inductor-capacitor circuits.

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